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STABILIZATION OF LINEAR DYNAMICAL SYSTEMS BY OPTIMAL CONTROLS OF LINEAR-QUADRATIC PROBLEMS†

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A method of obtaining a bounded feedback which stabilizes a linear dynamical system is described. The attainment of a positional solution of a special auxiliary linear-quadratic problem of optimal control using an optimal regulator is the basis of the method. The results are illustrated for the well-known problem of the stabilization of a pendulum in the upper unstable position by means of inertial controls. © 1998 Elsevier Science Ltd. All rights reserved.

It has been established [1, 2] that the optimal feedback-type control for the linear-quadratic problem of the construction of regulators with an infinite horizon is stabilizing feedback. The simplicity of the synthesis of a feedback-type control for a linear-quadratic problem with a finite horizon suggests the use of these solutions to stabilize unstable linear systems [3, 4]. It has been found that controls which minimize quadratic functions on the trajectories of non-linear systems also possess stabilizing properties [5]. The principal difference in the papers [3–5] lies in the fact that, while it is proposed in [3, 4] that well-known solutions of a linear-quadratic problem with a finite horizon are used as the stabilizing controls, only the stabilizing properties of feedback are proved in [5], without indicating any method of constructing the stabilizing control.

The main results in this paper are a proof of a single principle for constructing stabilizing feedbacks, a method for designing stabilizers which are constructed in accordance with this principle and an investigation of cases when the principle of sliding control ensures the asymptotic stability of the terminal set. Some analogies between feedback theorems in the theory of stability which guarantee the existence of Lyapunov functionals and practical methods for constructing such functions are noted. The result presented in this paper can be treated as a practical method of constructing Lyapunov functions using optimal control theory. Here, the emphasis is not placed on obtaining explicit expressions for the above-mentioned functions but rather on the possibility of the effective use of implicit expressions for the functions by appropriate computer calculations. In this approach, constructions in real time, which is natural when investigating real processes, play a decisive role.

By virtue of the specific details of the optimal control problems which are used, direct (geometric) constraints on the control were not taken into account when solving stabilization problems [1–5]. These were taken into account in the stabilization methods in [6, 7], on the basis of which special linear optimal control problems and a method of obtaining the optimal feedback using regulators [8, 9] were formulated.

The aim of this paper is to describe a method for stabilizing linear dynamical systems using regulators (stabilizers) which produce optimal feedback in a special linear-quadratic optimal control problem. The main difference from the approach which has previously been adopted [1–5], where a linear-quadratic optimal control problem was also used, is the fact that direct (geometric) constraints on the control are taken into account here.

1. FORMULATION OF THE PROBLEM

Suppose the behaviour of a dynamical system with a control in the interval $t \geqslant 0$ is described by the equation

$$\dot{x} = Ax + bu, \quad x(0) = x_0 \neq 0 \tag{1.1}$$

$$(x \in R^n, \quad u \in R; \ \operatorname{rank}(b, Ab, \dots, A^{n-1}b) = n)$$

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where $x = x(t)$ is the state of the system at the instant of time t, $u = u(t)$ is the value of the control action, and A and b are a constant matrix and a constant vector of corresponding dimensions.

We shall assume that the system is unstable when the control is disconnected ($u(t) \equiv 0, t \geqslant 0$). It is well known that the classical stabilization problem consists of finding a feedback $u = u(x), x \in R^n$ ($u(0) = 0$) such that system (1.1), which is closed by it

$$\dot{x} = Ax + bu(x), \quad x(0) = x_0 \tag{1.2}$$

possesses the following properties:

1. there are solutions which are extended to $t \geqslant 0$ in the neighbourhood of the equilibrium state, $x = 0$;
2. its zeroth solution is asymptotically stable in the Lyapunov sense.

In many applications only bounded controls are permissible, due to technical requirements. Hence, additional direct constraints $|u(x)| \leqslant L$ ($0 < L < \infty$) are imposed on the stabilizing controls in modern formulations of stabilization problems.

The problem of constructing a stabilizing feedback of the following type is investigated below, taking account of this constraint. Suppose that $G \subset R^n$ is a certain neighbourhood of the equilibrium state $x = 0$ of system (1.1) and that L, $0 < L < \infty$ is a specified number. We call the function

$$u(x), \quad x \in G \quad (u(0) = 0) \tag{1.3}$$

a (bounded) stabilizing feedback for the dynamic system (1.1) if

1. the function (1.3) satisfies the constraint

$$|u(x)| \leqslant L, \quad x \in G \tag{1.4}$$

2. closed system (1.2) has a solution $x(t)$, $t \geqslant 0$ for all $x_0 \in G$;
3. system (1.2) is asymptotically stable in G.

It is well known that the constraints (1.4) are typical in the modern theory of optimal control. It is therefore natural to attempt to invoke optimal control methods to construct the required stabilizing feedback. A special optimal control problem is introduced in Section 2 for this purpose and the stabilizing feedback will subsequently be constructed using this problem.

2. THE ASSOCIATED OPTIMAL CONTROL PROBLEM

We will now select the parameter of the method θ, $0 < \theta < \infty$ and consider the following linear-quadratic optimal control problem

$$V(z) = \min \int_0^\theta \frac{1}{2} u^2(t)dt \tag{2.1}$$

$$\dot{x} = Ax + bu, \quad x(0) = z \tag{2.2}$$

$$x(\theta) = 0 \tag{2.3}$$

$$|u(t)| \leqslant L, \quad t \in T = [0, \theta] \tag{2.4}$$

Problem (2.1)–(2.4) differs from the problems used in [1–5] for the purposes of stabilization primarily in the existence of the direct constraint (2.4) in the equation. Such a constraint was taken into account in [6, 7], but the associated optimal control problems were linear.

We shall call the piecewise-continuous function (henceforth, $t \in T$ everywhere in Sections 2 and 3, unless otherwise stated) a permissible programme control of problem (2.1)–(2.4) if it satisfies the direct constraint (2.4), and the trajectory $x(t) = x(t \mid z)$ of system (2.2) corresponding to it falls, at a specified instant θ, on the origin of the system of coordinates which corresponds to the terminal condition (2.3). The permissible control

$$u^0(t) = u^0(t \mid z) \tag{2.5}$$

and the trajectory $x^0(t) = x^0(t \mid z)$ of system (2.2) corresponding to it are called the optimal programme control and trajectory if the quality criteria (2.1) attain a minimum value along them. We know [10] that a neighbourhood of the origin of coordinates G exists such that an optimal control (2.5) of problem (2.1)–(2.4) exists for all points $z \in G$.

Following the classical definition of a feedback-type optimal control, we shall call the function

$$u^0(z) = u^0(0 \mid z), \quad z \in G \tag{2.6}$$

the optimal feedback-type start control.

It is shown in this paper that the function (2.6) is a stabilizing feedback for system (1.1) (Section 5) and a method for obtaining it is described (Sections 3, 4, 6 and 7).

3. OPTIMAL PROGRAMME CONTROL OF THE ASSOCIATED OPTIMAL CONTROL PROBLEM

In the proposed method, the process of stabilization begins at the instant of time $t = 0$ with the programme solution $u^0(t) = u^0(t \mid x_0)$ (2.5) of problem (2.1)–(2.4) for the initial state $z = x_0$ and, subsequently, only a continuous correction is made to the programme solutions. Since, for the state $z = x_0$, problem (2.1–2.4) only contains *a priori* information, the programme solution when $z = x_0$ can be constructed up to the start of the stabilization process by a finite first-order method [11, 12],† for example.

We shall say that problem (2.1)–(2.4) is a simple problem if, when the ends $\underline{t}_i(z)$, $i \in P_0(z) = \{s^0(z) + 1, \ldots, p(z)\}$; $\bar{t}_i(z)$, $i \in P^0(z) = \{1, \ldots, s^*(z)\}$ of the quasisingular segments of the optimal control (2.5) belong to T, the conditions

$$\frac{\partial}{\partial t}(\varphi^0(t) - k_{i-1}(z))|_{t=\underline{t}_i(z)} \neq 0, \quad i \in P_0(z) \tag{3.1}$$

$$\frac{\partial}{\partial t}(\varphi^0(t) - k_i(z))|_{t=\bar{t}_i(z)} \neq 0, \quad i \in P^0(z)$$

are satisfied, where $\varphi^0(t) = \varphi^0(t \mid z)$ is a function defined by the relation

$$\varphi^0(t) = \psi^{0'}(t)b \tag{3.2}$$

and $\psi^0(t) = \psi^0(t \mid z)$ is the solution (the cotrajectory) of the conjugate system $\dot{\psi} = -A'\psi$, $\psi(\theta) = y$, which correspond to the optimal n-vector of the potentials $y = y(z)$ and, in addition

$$\begin{aligned}
&|\varphi^0(t)| \leq L, \quad t \in T_i^0(z) = [\underline{t}_i(z), \bar{t}_i(z)] \\
&i \in P(z) = \{1, \ldots, p(z)\} \\
&|k_i(z)| = L, \quad i \in P_*(z) = \{s^0(z), \ldots, s^*(z)\} \\
&\varphi^0(\underline{t}_i(z)) = k_{i-1}(z), \quad u^0(t) \equiv k_{i-1}(z), \quad t \in T^-(\underline{t}_i(z)), \quad i \in P_0(z) \\
&\varphi^0(\bar{t}_i(z)) = k_i(z), \quad u^0(t) \equiv k_i(z), \quad t \in T^+(\bar{t}_i(z)), \quad i \in P^0(z)
\end{aligned} \tag{3.3}$$

where $T^-(t)$ is a small left-side neighbourhood of the point t, $T^+(t)$ is a small right-side neighbourhood and the prime indicates the operation of transposition (the values of the terms $s^0(z)$ and $s^*(z)$ are determined below).

It is well known [11, 12] that the optimal programme control $u^0(t) = u^0(t \mid x_0)$, which is continuous with respect to t, has the form

$$\begin{aligned}
&u^0(t) = -L, && \text{if} \quad \varphi^0(t) < -L \\
&u^0(t) = L, && \text{if} \quad \varphi^0(t) > L \\
&u^0(t) = \varphi^0(t), && \text{if} \quad |\varphi^0(t)| \leq L
\end{aligned} \tag{3.4}$$

According to (2.6), to construct the stabilizing control of system (1.1) for the current state $x(\tau)$ at an

†See also: LUBOCHKIN, A. V., Methods of solving convex optimal control problems. Candidate dissertation. Izd. Belarus. Gos. Univ., Minsk, 1987.

arbitrary current instant of time $\tau \geqslant 0$, it is necessary to know the solution of the associated problem (2.1)–(2.4) when $z = x(\tau)$. But we will not solve this problem explicitly now. The essence of the new approach is as follows: The optimal programme control $u^0(t \mid x(\tau))$ for the state $z = x(\tau)$ has the same form (3.4) as for the initial state $z = x_0$. This control is defined by the set

$$\begin{aligned} &\underline{t}_i(\tau) = \underline{t}_i(x(\tau)), \quad i \in P_0(\tau) = P_0(x(\tau)) \\ &\bar{t}_i(\tau) = \bar{t}_i(x(\tau)), \quad i \in P^0(\tau) = P^0(x(\tau)) \\ &y(\tau) = y(x(\tau)) \end{aligned} \tag{3.5}$$

which consists of the ends of the quasisingular segments belonging to the interval T, $T_i^0(\tau) = T_i^0(x(\tau))$, $i \in P(\tau) = P(x(\tau))$ (3.3) and the vector of the potentials. Here, $s^0(\tau) = s^0(x(\tau)) = 0$ if $\underline{t}_1(\tau) > 0$; $s^0(\tau) = 1$ if $\underline{t}_1(\tau) \leqslant 0 \leqslant \bar{t}_1(\tau)$; $s^*(\tau) = s^*(x(\tau)) = p(x(t)) = p(\tau)$ if $\bar{t}_{p(\tau)}(\tau) < \theta$; $s^*(\tau) = p(\tau) - 1$ if $\underline{t}_{p(\tau)}(\tau) \leqslant \theta \leqslant \bar{t}_{p(\tau)}(\tau)$.

Hence, to construct the control $u^0(0 \mid x(\tau))$, $\tau \geqslant 0$, it is sufficient to have the elements of (3.5) at each instant of time $\tau \geqslant 0$. Equations which describe the behaviour of the elements of (3.5) are derived in Section 6 for this purpose and a method for solving these equations is proposed in Section 7.

4. A STABILIZER

We will now assume that the optimal feedback (2.6) has been constructed. We close system (1.1) with it and consider the behaviour of the closed system

$$\dot{x} = Ax + bu^0(x) \tag{4.1}$$

for the actual initial state

$$x(0) = x_0^* \in G \tag{4.2}$$

We will denote by $x^*(t)$, $t \geqslant 0$ the solution of Eq. (4.1) which corresponds to it.

It can be seen that the function

$$u^*(t) = u^0(x^*(t)), \quad t \geqslant 0 \tag{4.3}$$

is applied to the input of the system during the stabilization process, that is, the optimal feedback (2.6) is not fully used and only its values along the actual trajectory $x^*(t)$, $t \geqslant 0$ of system (4.1) are required. Furthermore, there is no need to know the value of $u^0(x^*(\tau))$ in advance, it being sufficient to have this value only at the actual instant of time τ when system (4.1) is in the actual state $x^*(\tau)$.

Next, we take a further three facts into consideration: (1) in many modern control problems, which make use of computer techniques, the control actions are not fed continuously into the input of the controlled system but at discrete instants of time with a certain time step, (2) the rate of actual processes is finite, (3) modern computer hardware has extremely high-speed devices available. Taking account of these facts, it is shown below that, in the case of many practical processes, existing computers enable one to realize the optimal feedback (2.6).

We call function (4.3) the realization of the optimal feedback-type control corresponding to the initial state (4.2) and we will call the family of piecewise-continuous functions

$$u^*(h,t), \quad t \geqslant 0, \quad h \to 0 \tag{4.4}$$

of the form $u^*(h, t) \equiv f_s(t)$, $t \in [sh,(s + 1)h[$ $(s = 0, 1, \ldots)$, where $f_s(t)$ is a function which is known up to the instant of time, $t = sh$, the discrete realization of optimal feedback in the closed system (4.1) with the initial state (4.2), if, for any τ, $0 < \tau < \infty$

$$\int_0^\tau |u^*(t) - u^*(h,t)| dt \to 0 \quad \text{as} \quad h \to 0$$

For the chosen operating cycle $h > 0$, we shall call the element $u^*(h, t)$, $t \geqslant 0$ of family (4.4), the h-realization of the optimal feedback for the initial state (4.2).

Definition. We call any device, which, for each fixed sufficiently small number $h > 0$, is capable of calculating the value of the function $u^*(h, t)$, $t \geq 0$ in real time, a stabilizer.

The concept of "a solution in real time" is explained in Section 7.

In order to keep the discussion brief, we shall subsequently omit the dependence of the stabilizer on the selected parameter $h > 0$.

Hence, the problem of working out the optimal feedback has been reduced to describing an algorithm for the operation of the stabilizer (Section 7).

5. THE STABILITY OF THE CLOSED SYSTEM

Theorem. System (1.1), which is closed by the optimal feedback (2.6), is asymptotically stable in G.

Proof. Suppose the closed system (4.1) is in the state $x^*(\tau)$ at an arbitrary instant of time $\tau = sh(s = 0, 1, \ldots)$. In this state, the quality criterion of the auxiliary problem (2.1)–(2.4) takes the value $V(x^*(\tau))$. We shall show that $V(x)$, $x \in G$ is a Lyapunov function. It is clear that it is continuous and that $V(0) = 0$, $V(x) > 0$ when $x \neq 0$. We will calculate its value at the point $x = x^*(\tau + h)$. For $x^*(\tau + h)$, the value of $V(x^*(\tau + h))$ is determined by the solution of problem (2.1)–(2.4) when $z = x^*(\tau + h)$. This problem is equivalent to the following

$$\int_{\tau+h}^{\tau+h+\theta} \frac{1}{2} u^2(t)dt \to \min, \quad \dot{x} = Ax + bu$$

$$x(\tau + h) = x^*(\tau + h), \quad x(\tau + h + \theta) = 0$$

$$|u(t)| \leq L, \quad t \in [\tau + h, \tau + h + \theta]$$

The last problem, in turn, is equivalent to the problem

$$\int_{\tau}^{\tau+h} \frac{1}{2}(u^0(t|x^*(\tau)))^2 dt + \int_{\tau+h}^{\tau+h+\theta} \frac{1}{2} u^2(t)dt \to \min \tag{5.1}$$

$$\dot{x} = Ax + bu, \quad x(\tau) = x^*(\tau), \quad x(\tau + h + \theta) = 0$$

$$u(t) \equiv u^0(t \mid x^*(\tau)), \quad t \in [\tau, \tau + h[; \quad |u(t)| \leq L, \quad t \in [\tau + h, \tau + h + \theta]$$

Since the control $u(t) \equiv u^0(t|x^*(\tau))$, $t \in [\tau, \tau + \theta]$; $u(t) \equiv 0$, $t \in [\tau + \theta, \tau + h + \theta]$ is permissible in problem (5.1), the minimum value of the quality criterion of this problem does not exceed $V(x^*(\tau))$

$$\int_{\tau}^{\tau+h} \frac{1}{2}(u^0(t|x^*(\tau)))^2 dt + V(x^*(\tau + h)) \leq V(x^*(\tau)) \tag{5.2}$$

Since, under the assumptions which have been made, the optimal control $u^0(t|x^*(\tau))$, $t \geq \tau$, t cannot identically vanish in any interval of finite length, the integral in (5.2) is positive and, consequently, $V(x^*(\tau + h)) < V(x^*(\tau))$.

Next, it can be shown using arguments which are typical in the Lyapunov theory of stability [13, 14] that $V(x(^*sh)) \to 0$ as $s \to \infty$, $x^*(sh) \to 0$ as $s \to \infty$, from which it is easily shown that $x^*(t) \to 0$ when $t \to \infty$.

Remark. In the proof of the theorem, the discussion has actually not been about feedback (2.6) but rather about its discrete approximation, which is introduced using problem (2.1)–(2.4) like feedback (2.6) but using piecewise-continuous functions with a quantization period $h > 0$. The stabilizing property of feedback (2.6) can also be proved, but the reasoning is complicated in this case. What is more, only the above-mentioned discrete approximation is used in the actual stabilization (see below) of system (1.1). For sufficiently small $h > 0$ (which is assumed in this paper), the transients in the system, closed by the feedback, or its discrete approximation are practically indistinguishable (the proof of this fact is fairly standard and is not given in this paper).

6. THE DEFINING EQUATIONS OF THE STABILIZER

Suppose that problem (2.1)–(2.4) with $z = x^*(\tau)$ is simple in the case of the actual state $x^*(t)$ and that the equality

$$\mathrm{rank}(h(t),\ t \in T_0(\tau)) = n \tag{6.1}$$

holds, where $T_0(\tau) = \{t \in T: |\varphi^0(t)| < L\} = \cup\,]\underline{t}_i(\tau), \bar{t}_i(\tau)[, i \in P(\tau)$; $\varphi^0(t \mid x^*(\tau)), t \in T$ is the function (3.2), $h(t) = F(\theta - t)b, t \in T$, and $F(t), t \in T$ is the fundamental $n \times n$ matrix of the solutions of the system $\dot{x} = Ax$: $\dot{F} = AF, F(0) = E$. We introduce the notation

$$\underline{d}(t) = \partial(\varphi^0(t) - k_{i-1}(\tau)) / \partial t, \quad \bar{d}(t) = \partial(\varphi^0(t) - k_i(\tau)) / \partial t.$$

and construct the following numbers and sets: the ends of the quasisingular segments (3.5)

$$\underline{t}_i(\tau) < \bar{t}_i(\tau) < \underline{t}_{i+1}(\tau) < \bar{t}_{i+1}(\tau)$$
$$i \in P(\tau) \backslash \{p(\tau)\} \tag{6.2}$$

the terms $s^0(\tau), s^*(\tau)$ (see Section 3); $k_i(\tau) = -L$ if $\varphi^0(t) \leqslant -L$; $t \in T_i^*(\tau) = [\bar{t}_i(\tau), \underline{t}_{i+1}(\tau)]$, $k_i(\tau) = L$ if $\varphi^0(t) \geqslant L, t \in T_i^*(\tau), i \in P_*(\tau)$; $\underline{l}_*(0) = 0$ if $\underline{t}_1(\tau) > 0$, $\underline{l}_*(\tau) = 1$ if $\underline{t}_1(\tau) = 0$, $\bar{l}_*(\tau) = 0$ if $\bar{t}_1(\tau) > 0$, $\bar{l}_*(\tau) = 1$ if $\bar{t}_1(\tau) = 0$, $\bar{l}_*(\tau) = 0$ if $\bar{t}_{p(\tau)}(\tau) < \theta$, $\bar{l}_*(\tau) = 1$ if $\bar{t}_{p(\tau)}(\tau) = \theta$, $\underline{l}_*(\tau) = 0$ if $\underline{t}_{p(\tau)}(\tau) < \theta$, $\underline{l}_*(\tau) = 1$ if $\underline{t}_{p(\tau)}(\tau) = \theta$, $L(\tau) = \{i \in P_0(\tau): \underline{d}(\underline{t}_i(\tau)) = 0;\ i \in P^0(\tau): \bar{d}(\bar{t}_i(\tau)) = 0\}$ (we refer to the intervals $T_i^*(\tau)$, $i \in P_*(\tau)$ as being non-singular).

The set $S(\tau) = \{p(\tau), s^0(\tau), s^*(\tau); k_i(\tau), i \in P_*(\tau); l_*(\tau), \bar{l}_*(\tau); \underline{l}^*(\tau); \bar{l}_*(\tau); L(\tau)\}$ is called the structure of the optimal control. The structure is assumed to be non-degenerate if $\beta(\tau) = \underline{l}_*(\tau) + \bar{l}_*(\tau) + \underline{l}^*(\tau) + \bar{l}^*(\tau) + |L(\tau)| = 0$.

Suppose the structure $S(\tau) = S(\tau_0)$ is non-degenerate at the instant $\tau = \tau_0$. Using Cauchy's formula, from the joining conditions at the ends of the quasisingular intervals of the optimal control, which is continuous with respect to t, and the conditions for satisfying the terminal constraint (2.3), we obtain that, when $\tau = \tau_0$, the elements of (3.5) satisfy the equations

$$q(\underline{t}_i(\tau);\ k_{i-1};\ y(\tau)) = 0,\ \ i \in P_0;\ \ r(\bar{t}_i(\tau);\ k_i;\ y(\tau)) = 0,\ \ i \in P^0 \tag{6.3}$$
$$f(\underline{t}_i(\tau);\ i \in P_0;\ \bar{t}_i(\tau),\ \ i \in P^0;\ \ k_i,\ \ i \in P_*;\ y(\tau);\ x^*(\tau)) = 0$$

which are subsequently called the defining equations of the stabilizer.

Here

$$q(t; k_{i-1}; y) = y'h(t) - k_{i-1},\ \ i \in P_0$$
$$r(t; k_i; y) = y'h(t) - k_i,\ \ i \in P^0$$
$$f(\underline{t}_i, i \in P_0;\ \bar{t}_i,\ i \in P^0;\ k_i, i \in P_*; y; x) =$$
$$= F(\theta)x + \sum_{i=s^0}^{s^*} k_i \int_{\bar{t}_i}^{\underline{t}_{i+1}} h(t)dt + \sum_{i=1}^{p} \int_{\underline{t}_i}^{\bar{t}_i} h(t)h'(t)y dt$$

$p = p(\tau)$, $s^0 = s^0(\tau)$, $s_* = s_*(\tau)$; $k_i = k_i(\tau)$, $i \in P_* = P_*(\tau)\underline{t}_i = \underline{t}_i(\tau)$, $i \in P_0 = P_0(\tau)$; $t_i = \bar{t}_i(\tau)$, $i \in P^0 = P^0(\tau)$; $\bar{t}_0 = 0$, if $s^0 = 0$; $\underline{t}_1 = 0$, if $s^0 = 1$; $\underline{t}_{p+1} = \theta$, if $s^* = p$; $\bar{t}_p = \theta$, if $s_* = p - 1$

For system (6.3), we calculated the Jacobian matrix $H(\tau) = H(\underline{t}_i, i \in P_0; \bar{t}_i, i \in P^0; y)$, which consists of the blocks

$$H_{11}(\underline{t}_i, i \in P_0; y) = \mathrm{diag}(-y'\mu(\underline{t}_i),\ i \in P_0),\ \ H_{13}(\underline{t}_i, t \in P_0) = (h(\underline{t}_i), i \in P_0)'$$
$$H_{22}(\bar{t}_i, i \in P^0; y) = \mathrm{diag}(-y'\mu(\bar{t}_i),\ i \in P^0),\ \ H_{23}(\bar{t}_i, t \in P^0) = (h(\bar{t}_i), i \in P^0)'$$
$$H_{33}(\underline{t}_i, i \in P_0;\ \bar{t}_i, i \in P^0) = \sum_{i=1}^{p} \int_{\underline{t}_i}^{\bar{t}_i} h(t)h'(t)dt$$
$$(\mu(t) = F(\theta - t)Ab,\ t \in T)$$

(the remaining blocks are null matrices).

It can be shown that, since $\beta(\tau_0) = 0$ and relations (3.1) and (6.1) are satisfied, then $\det H(\tau_0) \neq 0$. According to the Implicit Function Theorem, it follows from this that, for $\tau \in T^+(\tau_0)$, a unique continuous solution (3.5) of Eqs (6.3) exists (here, $S(\tau) = S(\tau_0)$). A method of solving Eqs (6.3) is described in Section 7.

7. AN ALGORITHM FOR THE OPERATION OF THE STABILIZER

Before describing the algorithm for the operation of the stabilizer, we will present a numerical method for solving its defining equations (6.3).

We consider the interval $]\tau_0, \tau_1[$, $\tau_1 > \tau_0 \geq 0$ for which $\beta(\tau) = 0$, $\tau \in]\tau_0, \tau_1[$. We assume that the starting values $p(\tau_0)$, $s^0(\tau_0)$, $s^*(\tau_0)$; $k_i(\tau_0)$, $i \in P_*(\tau_0)$; $\underline{t}_i = \underline{t}_i(\tau_0)$, $\bar{t}_i = \bar{t}_i(\tau_0)$, $i \in P^0(\tau_0)$; $y(\tau_0)$, are known for $\tau = \tau_0$ and describe the simplest algorithm (which uses Newton's method) for solving the finite equations (6.3) for $\tau \in]\tau_0, \tau_1[$, where $p = p(\tau_0)$, $s^0 = s^0(\tau_0)$, $s^* = s^*(\tau_0)$; $k_i = k_i(\tau_0)$, $i \in P_* = P_*(\tau_0)$; $P_0 = P_0(\tau_0)$, $P^0 = P^0(\tau_0)$.

As in the numerical solution of ordinary differential equations, we shall construct an approximate solution of Eqs (6.3) in the mesh $\tau_0, \tau_0 + h, \dots, \tau_0 + N_0 h = \tau_1$, where $h > 0$ is a specific parameter. We will initially assume that the starting conditions are such that conditions (6.2) are satisfied. In this case, by virtue of assumptions (6.1) and $\beta(\tau_0) = 0$, we have $\det H(\tau_0) \neq 0$.

Suppose that the sequence of solutions

$$\begin{aligned}&\omega(\tau_0 + sh) = (\underline{t}_i(\tau_0 + sh), i \in P_0;\ \bar{t}_i(\tau_0 + sh),\ i \in P^0\\ &y(\tau_0 + sh)),\quad s = 0, 1, \dots, \upsilon - 1\end{aligned}$$

which corresponds to the sequence of states $x^*(\tau_0 + sh)$, $s = 0, 1, \dots, \upsilon - 1$ is constructed, and that, for $s = 0, 1, \dots, \upsilon - 1$, the equalities

$$\begin{aligned}&f(\underline{t}_i(\tau_0 + sh),\ i \in P_0;\ \bar{t}_i(\tau_0 + sh),\ i \in P^0;\ k_i, i \in P_*; y(\tau_0 + sh);\ x^*(\tau_0 + sh)) = 0\\ &q(\underline{t}_i(\tau_0 + sh); k_{i-1};\ y(\tau_0 + sh)) = 0,\ \ i \in P_0\\ &r(\bar{t}_i(\tau_0 + sh); k_i;\ y(\tau_0 + sh)) = 0,\ \ i \in P^0\end{aligned}$$

are satisfied with a specified accuracy.

To calculate the solution $\omega(\bar{\tau})$, $\bar{\tau} = \tau_0 + \upsilon h$ in the new state $x^*(\tau_0 + \upsilon h)$, $\upsilon \leq N_0$ we construct the vectors $\omega^l = (\underline{t}_1^l, i \in P_0; \bar{t}_i^l, i \in P^0; y^l)$ $(l = 1, 2, \dots, l_0)$

$$\begin{aligned}&\omega^1 = (\underline{t}_i^1 = \underline{t}_i(\bar{\tau} - h),\ \ i \in P_0;\ \ \bar{t}_i^1 = \bar{t}_i\ (\bar{\tau} - h),\ \ i \in P^0; y^1 = y(\bar{\tau} - h))\\ &\omega^l = \omega^{l-1} - H^{-1}(\underline{t}_i^{l-1},\ \ i \in P_0;\ \bar{t}_i^{l-1},\ \ i \in P^0;\ \ y^{l-1})(q(\underline{t}_i^{l-1}; k_{i-1}; y^{l-1}),\\ &i \in P_0;\ (r(\bar{t}_i^{l-1}; k_i; y^{l-1}),\ \ i \in P^0;\ \ f'(\underline{t}_i^{l-1}, i \in P_0;\ \bar{t}_i^{l-1}, i \in P^0;\\ &k_i,\ \ i \in P_*;\ \ y^{l-1};\ \ x^*(\bar{\tau})))',\ \ l = 2, 3, \dots, l_0\end{aligned} \tag{7.1}$$

and put

$$\omega(\bar{\tau}) = \omega^{l_0}:\ \underline{t}_i(\bar{\tau}) = \underline{t}_i^{l_0},\ \ i \in P_0;\ \ \bar{t}_i(\bar{\tau}) = \bar{t}_i^{l_0},\ \ i \in P^0;\ \ y(\bar{\tau}) = y^{l_0}$$

By virtue of the well-known properties of Newton's method [15], we have: for any $\varepsilon > 0$, sufficiently small $h > 0$ and sufficiently large $l_0 = l_0(h)$, functions $\underline{t}_i(\tau_0 + sh)$, $i \in P_0$; $\bar{t}_i(\tau_0 + sh)$, $i \in P^0$; $y(\tau_0 + sh)$, $s = 0, 1, \dots, N_0$ can be constructed such that, for $s = 0, 1, \dots, N_0$, the inequalities

$$\begin{aligned}&|q(\underline{t}_i(\tau_0 + sh); k_{i-1};\ y(\tau_0 + sh))| \leq \varepsilon,\ \ i \in P_0\\ &|r(\bar{t}_i(\tau_0 + sh); k_i;\ y(\tau_0 + sh))| \leq \varepsilon,\ \ i \in P^0\\ &\|f(\underline{t}_i(\tau_0 + sh),\ \ i \in P_0;\ \bar{t}_i(\tau_0 + sh),\ \ i \in P^0;\ k_i, i \in P_*;\ y(\tau_0 + sh);\ \ x^*(\tau_0 + sh))\| \leq \varepsilon\end{aligned}$$

are satisfied.

We will now consider different cases of degeneracy of the structure of the control. An additional quasisingular (non-singular) segment can occur in the interval T at the instant of time $\tau = \tau_1$ through its left end. The instant of entry τ_1 can be found, following the value of $\varphi^0(0)$. If a quasisingular segment is added, then, when $\tau > \tau_1$, in Eqs (6.3), we replace p by $\bar{p} = p + 1$, s^0 by $\bar{s}^0 = s^0 + 1$ and s^* by $\bar{s}^* = s^* + 1$ and renumber the points t_{-i}, $i \in \bar{P}_0$; $\bar{t}_i$, $i \in \bar{P}^0$ and the values k_i, $i \in \bar{P}_*$. If a non-singular segment is added, we replace s^0 by $\bar{s}^0 = s^0 - 1$ and determine the value of $\bar{k}_0$, using the value of $\varphi^0(0)$. In the rules for the solution of the defining equations (6.3), we only change the rule for calculating the vector ω^1 (7.1): in the first case, we put $\bar{t}^1{}_i = h$ for the components $\bar{t}^1{}_i$ and, in the second case, we put $\underline{t}_1^1 = h$ for $\underline{t}_1^1$. The quasisingular (non-singular) segment existing in T can leave through the left segment of T. The instant of departure τ_1 is found using the value of $\bar{t}_1^1(\tau)(\underline{t}_1(\tau))$. If the quasisingular segment departs, then we put $\bar{p} = p - 1$, $\bar{s}^0 = s^0 - 1$, $\bar{s}^* = s^* - 1$ and we renumber the points $\underline{t}_i$, $i \in \bar{P}_0$; $\bar{t}_i^1$, $i \in \bar{P}^0$ and the values k_i, $i \in \bar{P}^*$. If a non-singular segment departs, we put $\bar{s}^0 = s^0 + 1$.

We deal with similar situations at the right-hand end of the interval T in an analogous manner. The appearance or disappearance of intervals within T is also possible. These situations are dealt with in a similar way to that adopted previously in [8, 9].

The algorithm for the operation of the stabilizer involves the following: when $t \in [0, h[$, $h > 0$, the stabilizer uses the solution (2.5) of problem (2.1)–(2.4) for $z = x_0$ (this solution can be constructed earlier prior to the connection of the stabilizer): $u^*(t) \equiv u^0(t \mid x_0)$, $t \in [0, h[$. The algorithm for the operation of the stabilizer when $t \geqslant h$ is made up, for each $[sh, (s + 1)h[$ $(s = 1, 2, \ldots)$, of the following operations: (1) for a known actual state $x^*(sh)$, the stabilizer, using the method described above, constructs the solution $\omega(sh)$ of Eqs (6.3) employing $\omega((s - 1)h)$ as the initial approximation; (2) in the interval $[sh, (s + 1)h[$, the stabilizer uses the equation $u^*(t) \equiv u^0(t - sh \mid x^*(sh))$, $t \in [sh, (s + 1)h[$.

On the basis of the well-known properties of Newton's method, it is easy to calculate the amount of work required to calculate the solution of the defining equations with a specified accuracy at an instant of time τ subject to the condition that the known solution for the instant of time $\tau - h$ is used as the initial approximation. If the available computer is capable of carrying out this work in a time not exceeding h units of real time, it is natural to assume that Eqs (6.3) are solved under real-time conditions,

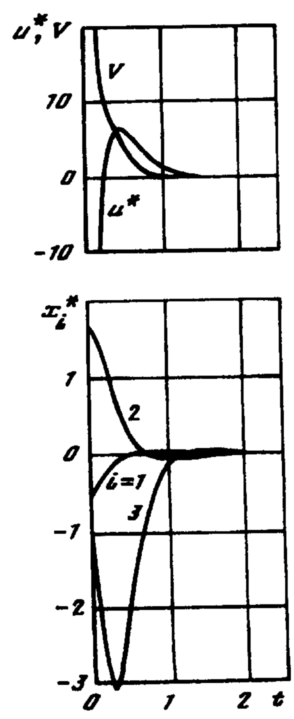

Fig. 1.

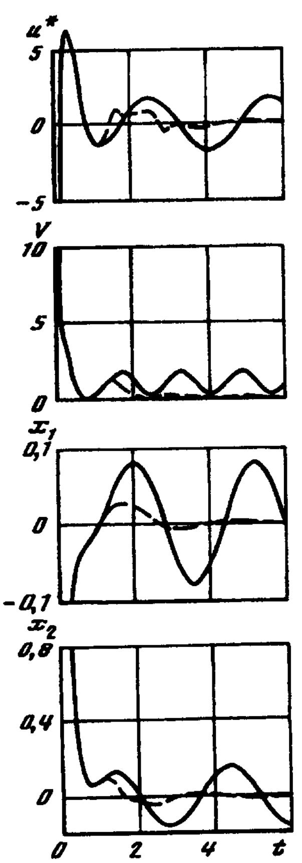

Fig. 2.

and this means that the stabilizer constructs the realization of the feedback at the same rate. Clearly, the attainment of such a condition depends on the complexity of the system under consideration and on the power of the computers used.

8. EXAMPLE

As an illustration, consider the problem [16] of stabilizing a pendulum in the unstable upper position of equilibrium using a moment applied to it on the suspension axis. This moment is worked out by a control mechanism which is an integrating circuit. The control mechanism, in its turn, is subject to a certain control action, u. The linearized equations of motion have the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 + x_3, \quad \dot{x}_3 = u \tag{8.1}$$

Plots of the implementation of the feedback $u^*(t) = u^0(x^*(t))$, $t \geqslant 0$, worked out by the stabilizer, the change in time of the Lyapunov function $V(x^*(t))$, $t \geqslant 0$ when $h = 0, 01$, $\theta = 1$, $L = 10$, and graphs of the corresponding components of the trajectories $x^*_i(t)$, $t \geqslant 0$, $i = 1, 2, 3$ are shown in Fig. 1.

In the numerical experiment, the operator of the stabilizer was tested with a perturbation acting on the system. A system was considered which differed from (8.1) in that there was a term $-\cos 2t$ in the second equation. Graphs of the functions $u^*(t)$, $V(x^*(t))$, $x_1(t)$, $x_2(t)$ are represented by the solid lines in Fig. 2. The performance of the transients depends very much on the magnitude of the parameter θ. On account of this, an experiment was conducted using the following rule for the change in the parameter θ in the stabilization process. Three numbers were selected: $\theta^* = 1$, $\theta_* = 0.4$, $h_\theta = 0.005$. Initially, we put $\theta = \theta^*$. If it turned out that $V(x^*(t)) > V(x^*(t-h))$, the value of θ was reduced by an amount h_θ (here, in the case when $\theta = \theta_*$, the magnitude of the parameter θ remained unchanged). The results obtained are represented by the dashed line in Fig. 2.

It is seen from these graphs that control with a parameter θ can turn out to be an effective means of increasing the effectiveness of the stabilization under permanent perturbations.

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